KNOT INVARIANTS

by

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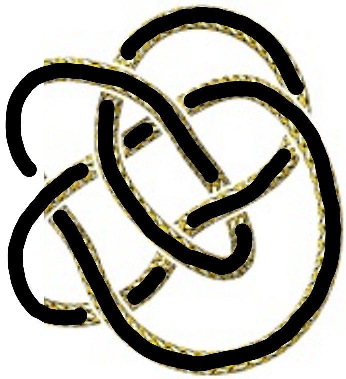
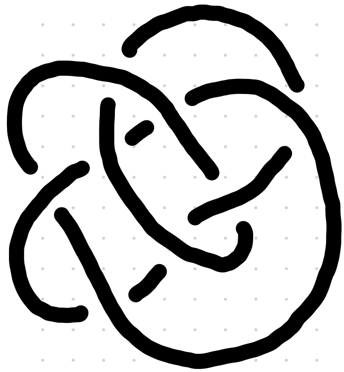
website with this packet and more fun math!

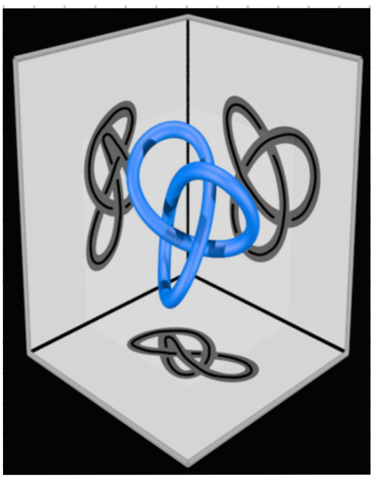
DIAGRAMS

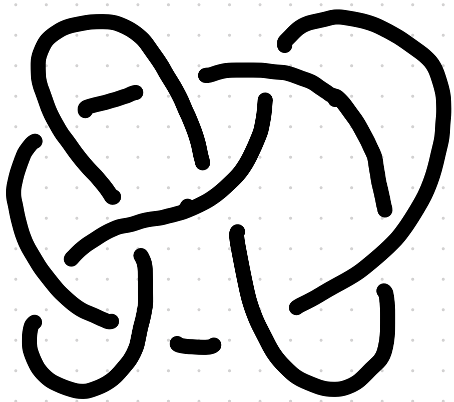
To draw knots on paper, we can take a picture of them, and record the crossings in a diagram:





Because you can rearrange the parts of a knot, it may have very different looking diagrams:





This looks knotted, but isn't!

It turns out that any two diagrams of the same knot are connected by something called Reidemeister moves. However, we need a simpler way of telling knots apart.

INVARIANTS

Invariants are a way of describing a knot that doesn't change as you manipulate the knot: they are the same for all diagrams of a knot. Because of this, if two diagrams have different invariants, they must represent different knots.

There are many invariants, here are some of the simplest:

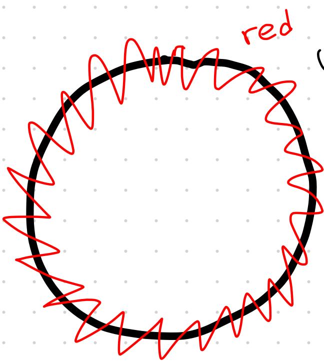
Tricolorability

the curves between crossings

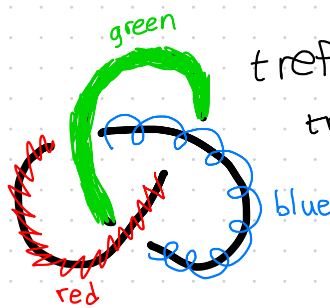
A knot is tricolorable if you can color the arcs of any diagram with three colors, following two rules:

1. At any crossing, either just one, or all three colors are used.
2. At least two colors are used.

For example, the unknot is not tricolorable but the trefoil is, so they are different knots.



unknot is not tricolorable



trefoil is tricolorable

No, only one color used.

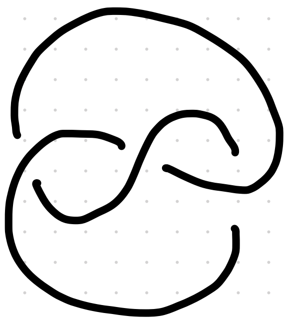
Yes, both rules work.

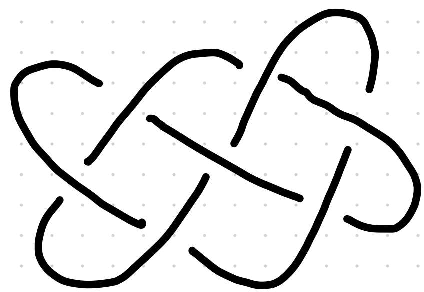
Since one is tricolorable and the other is not, these must be different knots!

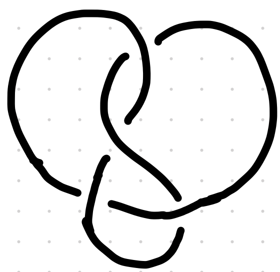
Note: you can tricolor one diagram of a knot if and only if you can tricolor them all, so you you only need to check one diagram.

Try it yourself!

Which of the following knot diagrams can be tricolored? Based on this, which could be the unknot?





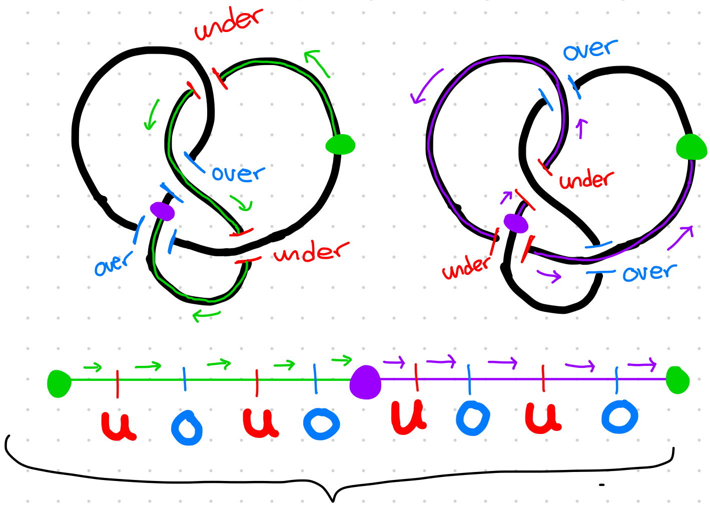


One of the above is not tricolorable, but using our next invariant, we'll see it is not the unknot. This shows that invariants can't always tell knots apart.

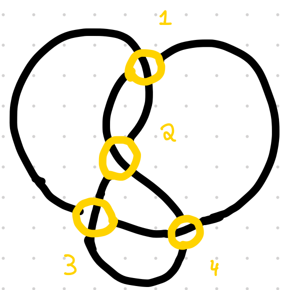
Crossing Number

The crossing number of a knot is the smallest number of crossings in any diagram of the knot. But this means we have to check all possible diagrams!

Peter Tait showed that for an alternating diagram, this is just the number of crossings. A diagram is alternating if the crossings alternate over and under as you go along the diagram.

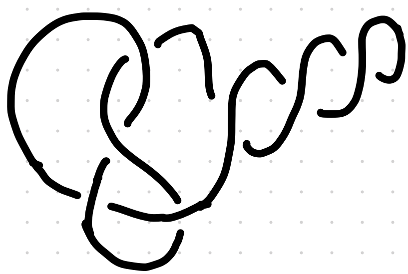


The diagram is alternating



There are four crossings, so the crossing number is four.

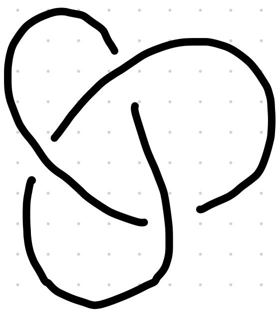
Note: if a diagram is not alternating, the crossing number of the knot may be less than the number of crossings in the diagram. This is because we could twist parts to add more crossings.

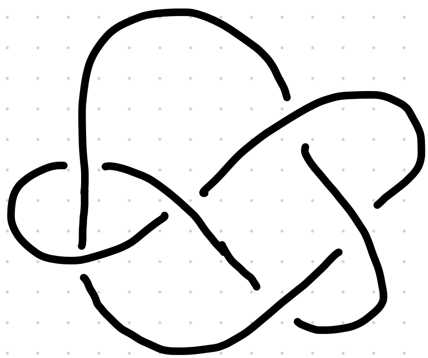


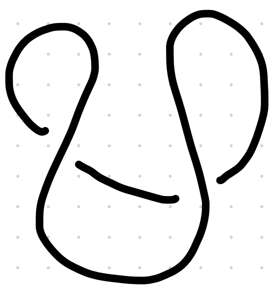
Some knots also have no alternating diagrams (whether or not they do is also an invariant!)

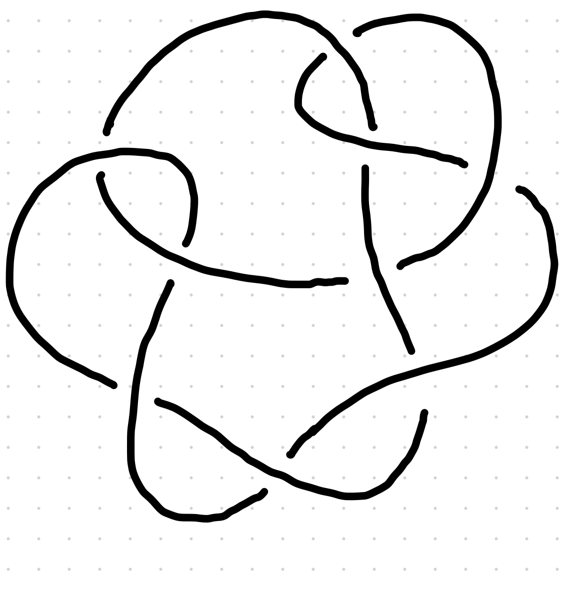
Try it yourself!

What are the crossing numbers of the following knots?









Hiearchy of Invariants

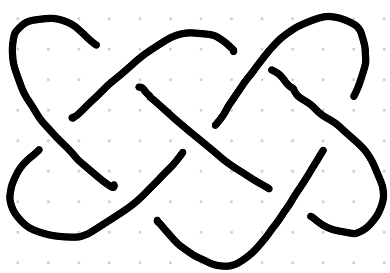
Many more invariants have been studied:

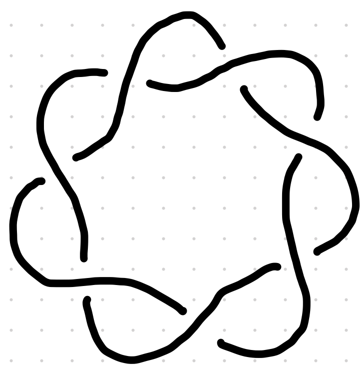
True/False	Tricolorability, Invertibility, Chirality ...
Number	Crossing #, Unknotting #, Stick # ...
Polynomial	Jones, Alexander, Conway, Kauffman ...
Group(s)	Fundamental Group of the complement, Khovanov Homology ...

⋮

In general, more complicated invariants give more information, so can tell more knots apart, but this isn't always the case.

The crossing number tells apart all the knots shown so far, but it cannot tell these two apart.



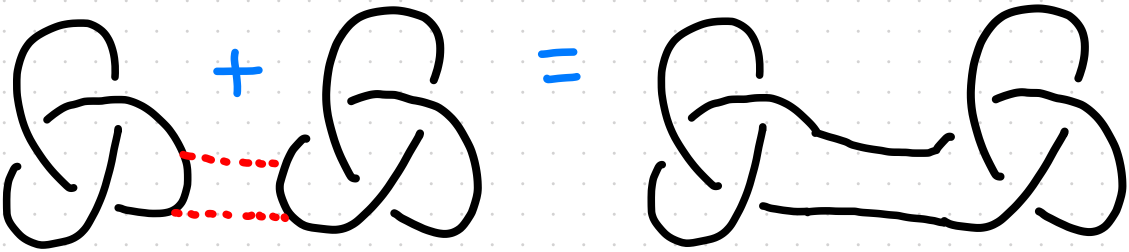


Surprisingly the simpler invariant of tricolorability can!

Try to explain why one is not tricolorable, and how this shows they aren't the same knot.

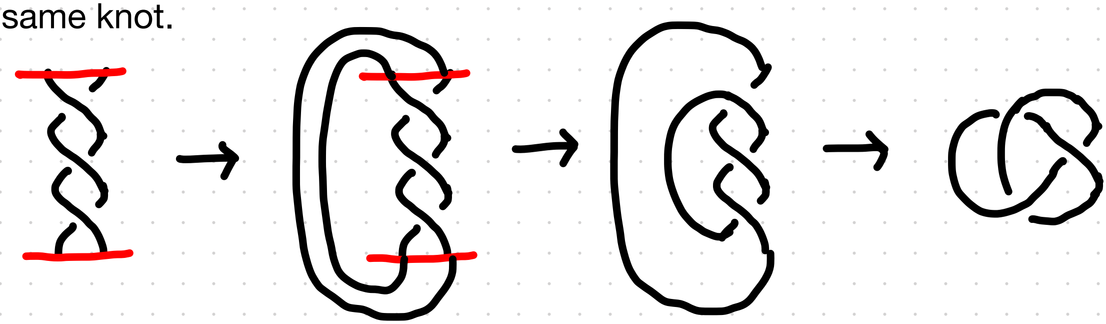
Bonus Fun Tidbits

Knots can be added by taking their connected sum. This allows one to break large knots into simpler parts. Knots that can't be divided in this way are called prime knots, like how prime numbers can't be divided.



Chiral knots are distinct from their mirror images. The trefoil is chiral, but the figure 8 knot is not. If a knot is not chiral, then the coefficients of its Jones polynomial are symmetric.

Alexander's theorem shows that every knot can be represented by a braid, if you connect its ends. More than one braid can represent the same knot.



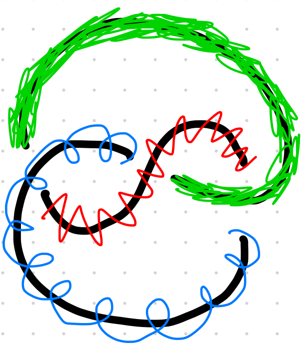
The Dowker code is a way to translate a diagram into a list of numbers. These can be enumerated to find every knot! Unfortunately it is not an invariant, though there are perfect invariants that tell apart every knot (they are too hard to compute though).

Check out "The Knot Book" by Colin Adams to learn more!

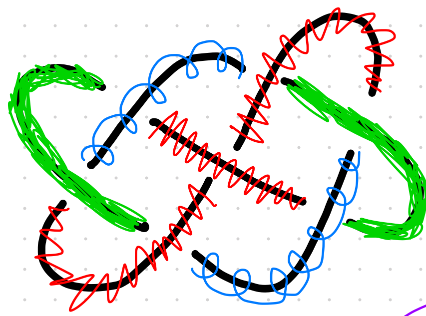
Try it yourself!

Solutions

Which of the following knot diagrams can be tricolored? Based on this, which could be the unknot?

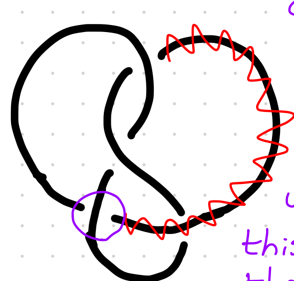


yes



yes

First we color one arc arbitrarily. Then, to satisfy rule 1 at the circled crossing, we can use

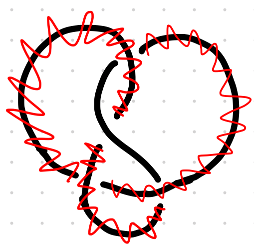


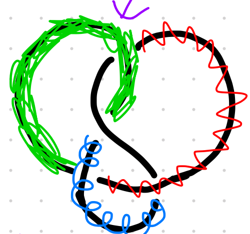
Either way fails, so we can't tri-color this not. We can't tri-color the unknot, so this may be the unknot.

all same

or

all three





Fail, to satisfy rule 1, the last arc must be red, but then rule 2 is violated

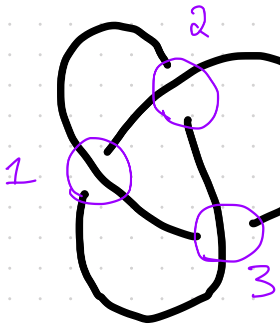
Fail, regardless of the color used, one crossing will only have two colors.

One of the above is not tricolorable, but using our next invariant, we'll see it is not the unknot. This shows that invariants can't always tell knots apart.

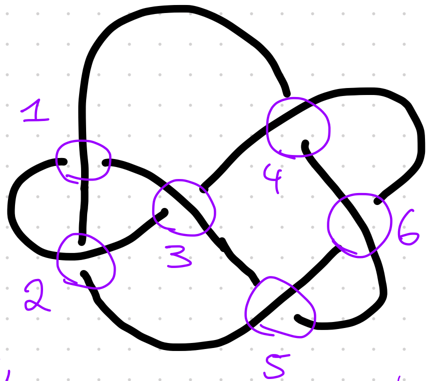
Try it yourself!

Solutions

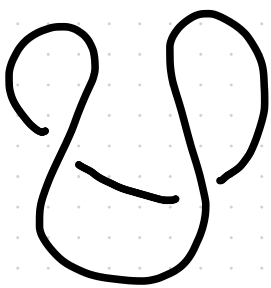
What are the crossing numbers of the following knots?



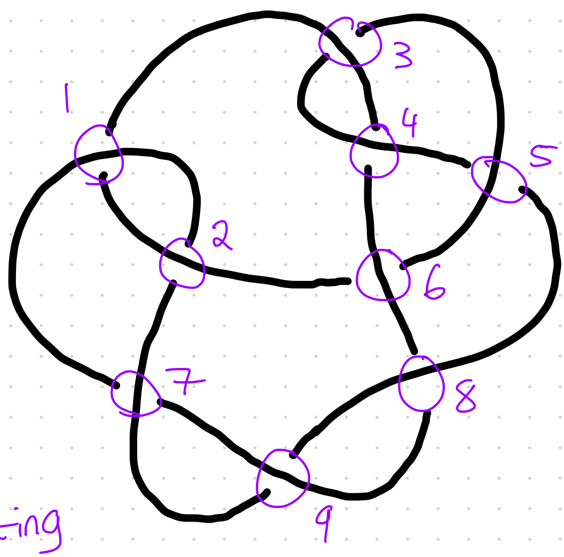
The diagram is alternating, so the crossing number is 3



The diagram is alternating, so the crossing number is 6



The diagram is <u>not</u> alternating. By untwisting the ends, we see it is the unknot, so has crossing number 0.



The diagram is alternating, so the crossing number is 9